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The nonperturbative propagator and vertex in massless quenched QED$_d$

A Bashir[1,2] **and R Delbourgo**[2]

[1] Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Apartado Postal 2-82, Morelia, Michoacán 58040, Mexico
[2] School of Mathematics and Physics, University of Tasmania, Hobart 7001, Australia

E-mail: adnan@itzel.ifm.umich.mx and Bob.Delbourgo@utas.edu.au

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Abstract

It is well known how multiplicative renormalizability of the fermion propagator, through its Schwinger–Dyson equation, imposes restrictions on the 3-point fermion–boson vertex in massless quenched quantum electrodynamics in four dimensions. Moreover, perturbation theory serves as an excellent guide for possible nonperturbative constructions of Green functions. We extend these ideas to arbitrary dimensions d. The constraint of multiplicative renormalizability of the fermion propagator is generalized to a Landau–Khalatnikov–Fradkin transformation law in d dimensions and it naturally leads to a constraint on the fermion–boson vertex. We verify that this constraint is satisfied in perturbation theory at the one-loop level in three dimensions. Based upon one-loop perturbative calculation of the vertex, we find additional restrictions on its possible nonperturbative forms in arbitrary dimensions.

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1. Introduction

The behaviour of the fermion propagator and the fermion–boson vertex in quantum electrodynamics (QED), along with higher Green functions, is dictated by the corresponding Schwinger–Dyson equations (SDE). However, one expects that the gauge theory conditions such as the Ward–Green–Takahashi identity (WGTI) [1][3], and the Landau–Khalatnikov–Fradkin transformations (LKFT) [2] (derived also by Johnson and Zumino through functional methods [3]), should impose simplifying constraints on the equations that these functions

[3] It was also given by Fradkin, see the corresponding reference in [2].

satisfy. As these conditions are nonperturbative in nature, one might hope to arrive at a better understanding of the propagator as well as the vertex to an arbitrary order in perturbation theory. The WGTI is relatively straightforward to implement and its constraints easier to understand. In contrast, extracting useful information from the LKFT has proved a harder nut to crack except only in a small number of special cases and even then success has been achieved to a limited extent.

The WGTI for the vertex determines what is often called its longitudinal part [4]. In the quenched version of four-dimensional QED (QED_4), it has long been known that the remaining transverse part plays a crucial role in ensuring the correct LKFT or, in other words, the multiplicative renormalizability of the fermion propagator [5, 6]. This consideration was taken into account in [7] and [8] to propose a transverse vertex. These *ansatz* ensure that the fermion propagator is multiplicatively renormalizable. However, the construction involves the assumption that the anomalous dimension γ is zero in the Landau gauge; crucially this is not the general solution, and disagrees with the perturbation theory beyond one loop [9, 10]. This was noted and fixed in [11]; there it was shown that a fermion propagator which obeys its LKFT at every order of the perturbation theory satisfies a simple equation in momentum space. Moreover, it governs a transverse part of the fermion–boson vertex which can be reduced just to one unknown function. In the same article, this function was evaluated at the one-loop level in the limit when momentum in one of the fermion legs is much greater than in the other, imposing further constraints on the possible forms of the vertex. However, this effort was restricted only to four dimensions. In this paper, we investigate how these arguments could be extended to arbitrary dimensions for the two- and three-point Green functions.

There exist some earlier works in that context [12, 13]. In the perturbation theory, these works use the restrictions imposed by the two-point Green function at the one-loop level to constrain the vertex. In the present paper, we pose the question whether the fermion propagator could satisfy a simple integral equation in momentum space in d dimensions just like the one in four dimensions proposed in [11] such that the gauge covariance is preserved not just to one loop but to all orders in the perturbation theory. It appears that the answer is in the affirmative. We propose such an equation, which happily reduces to the one in [11] for the special case of $d = 4$. Moreover, we verify that in three dimensions, it gives us the correct gauge covariance of the fermion propagator up to three loops. We also argue that the terms of the type $(\alpha\xi)^n$ in the fermion propagator governed by this equation will be correct in arbitrary dimensions to *every order in the perturbation theory*.

The SDE for the fermion propagator ties it to the fermion–boson vertex and imposes a direct constraint on the vertex. We verify that such a constraint is indeed satisfied in three dimensions at the one-loop level for the vertex. In addition to this restriction, we know that the perturbation theory plays a vital role in providing us with a guide to which every nonperturbative *ansatz* of the vertex reduce in the region of weak coupling. Whereas constructing such a vertex is highly non-trivial for all values of the fermion momenta involved, except in lower dimensions [14], it is more likely to be achieved for some asymptotic limits. We will largely focus on the limit when momentum in one of the fermion legs is much greater than the one in the other n of the Euclidean regime. Knowing the one-loop vertex in this limit is made possible by the recent perturbative calculation of the transverse vertex by Davydychev *et al* [15], carried out in an arbitrary covariant gauge in arbitrary dimensions. In the said limit, we find a neat relation between the transverse vertex and the fermion propagator, reminiscent of the one which exists between the longitudinal vertex and the fermion propagator. As a verification of our result, we see that for the particular cases of $d = 3$ and 4, we recover the results known in the literature.

2. The massless fermion propagator

The SDE for the unrenormalized fermion propagator $S(p)$, in quenched QED, with a bare coupling e is given by

$$S^{-1}(p) = \not{p} + \mathrm{i}e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \gamma^\mu S(k) \Gamma^\nu(k, p) \Delta^0_{\mu\nu}(q) \tag{1}$$

where $q = k - p$ and d is the dimension of space–time. For massless fermions, $S(p)$ can be expressed in terms of a single Lorentz scalar function, $F(p^2)$, associated with the wavefunction renormalization, so that $S(p) = F(p^2)/\not{p}$; the bare propagator has $F = 1$. The photon propagator remains unrenormalized in quenched QED and is given by $\Delta^0_{\mu\nu}(q) = (g_{\mu\nu} + (\xi - 1)q_\mu q_\nu/q^2)/q^2$, where ξ is the standard covariant gauge parameter. $\Gamma^\mu(k, p)$ is the full fermion–boson vertex for which we must make an *ansatz*. Ball and Chiu [4] considered the vertex as a sum of longitudinal and transverse components. i.e., $\Gamma^\mu(k, p) = \Gamma^\mu_L(k, p) + \Gamma^\mu_T(k, p)$, where $\Gamma^\mu_T(k, p)$ is defined by $q_\mu \Gamma^\mu_T(k, p) = 0$. To satisfy equation (1) in a manner free of kinematic singularities, which in turn ensures the Ward identity is fulfilled, we have a possible solution (following Ball and Chiu):

$$\Gamma^\mu_L(k, p) = \mathcal{A}(k^2, p^2)\gamma^\mu + \mathcal{B}(k^2, p^2)(\not{k} + \not{p})(k + p)^\mu \tag{2}$$

where

$$\begin{aligned} \mathcal{A}(k^2, p^2) &= \frac{1}{2}\left(\frac{1}{F(k^2)} + \frac{1}{F(p^2)}\right) \\ \mathcal{B}(k^2, p^2) &= \frac{1}{2}\left(\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right)\frac{1}{k^2 - p^2} \end{aligned} \tag{3}$$

and $\Gamma^\mu_T(p, p) = 0$. Ball and Chiu [4] demonstrated that a set of 8 vectors $T^\mu_i(k, p)$ formed a general basis (slightly modified later by Kızılersü *et al* [16]) for the transverse part. However, for the massless case, only four of the basis vectors contribute to the fermion propagator equation and they are the same in both the schemes. Therefore, we can write

$$\Gamma^\mu_T(k, p) = \sum_{i=2,3,6,8} \tau_i(k^2, p^2, q^2) T^\mu_i(k, p). \tag{4}$$

The condition $\Gamma^\mu_T(p, p) = 0$ is then satisfied provided that in the limit $k \to p$, the $\tau_i(p^2, p^2, 0)$ are finite. One can then define the Minkowski space basis in the massless case to be [4, 16]:

$$\begin{aligned} T^\mu_2(k, p) &= (p^\mu k \cdot q - k^\mu p \cdot q)(\not{k} + \not{p}) \\ T^\mu_3(k, p) &= q^2\gamma^\mu - q^\mu \not{q} \\ T^\mu_6(k, p) &= \gamma^\mu(k^2 - p^2) - (k + p)^\mu(\not{k} - \not{p}) \\ T^\mu_8(k, p) &= \gamma^\mu \sigma_{\lambda\nu} k^\lambda p^\nu - k^\mu \not{p} + p^\mu \not{k}. \end{aligned} \tag{5}$$

On multiplying equation (1) by $\not{p}$, taking the trace, making use of equations (2)–(5), and Wick rotating to Euclidean space, we obtain

$$\begin{aligned} \frac{1}{F(p^2)} = 1 - \frac{e^2}{p^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{F(k^2)}{k^2 q^2} \Big\{ & \mathcal{A}(k^2, p^2)\frac{1}{q^2}[-2\Delta^2 + (1 - d)q^2 k \cdot p] \\ & + \mathcal{B}(k^2, p^2)\frac{1}{q^2}[-2(k^2 + p^2)\Delta^2] - \frac{\xi}{F(p^2)}\frac{p^2}{q^2}(k^2 - k \cdot p) \\ & + \tau_2(k^2, p^2, q^2)[-(k^2 + p^2)\Delta^2] + \tau_3(k^2, p^2, q^2)[-2\Delta^2 + (1 - d)q^2 k \cdot p] \\ & + \tau_6(k^2, p^2, q^2)[(1 - d)(k^2 - p^2)k \cdot p] + \tau_8(k^2, p^2, q^2)[(d - 2)\Delta^2] \Big\} \end{aligned} \tag{6}$$

where $\Delta^2 = (k \cdot p)^2 - k^2 p^2$. If we knew what the $\tau_i(k^2, p^2, q^2)$ were, we could solve this equation for the fermion propagator. In practice, it is obviously not an easy task, since these transverse functions are only known in the perturbation theory or are connected to higher order Green functions by gauge identities and SDEs. It is conceivable that only a small part of the right-hand side of equation (6) is sufficient to yield the main characteristics of the full propagator. We look at this possibility in the following subsection, but first we note that $\mathcal{B}$ is of order e^2 like τ_i and both can only contribute to F at least to order e^4. The dominant term which is at least of order e^2 in F is associated with the ξ-piece.

2.1. The $(\alpha\xi)^n$ terms

Bashir and Pennington [8] noted that if one assumes the anomalous dimension vanishes in the Landau gauge in the quenched case, the third term on the right-hand side of equation (6) gives the total answer for the fermion propagator in four dimensions. To begin with, let us so truncate equation (6) for arbitrary dimensions as well to see where this gets us. We would then deduce

$$\frac{1}{F(p^2)} = 1 + \xi e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{F(k^2)}{F(p^2)} \frac{k \cdot q}{k^2 q^4}. \tag{7}$$

Thus in four dimensions, on carrying out the angular integration, one can immediately see that the resulting equation has the solution $F(p^2) = (p^2/\Lambda^2)^\nu$. Here $\nu = a\xi, a = \alpha/4\pi$ and $\alpha = e^2/4\pi$. The solution correctly embodies the leading gauge terms which are proportional to $\alpha\xi$, $(\alpha\xi)^2$, $(\alpha\xi)^3$, etc as is obvious in the expansion $F(p^2) = 1 + \nu \ln p^2/\Lambda^2 + (\nu^2/2!) \ln^2 p^2/\Lambda^2 + \cdots$. What happens in arbitrary dimensions? Let us assume that equation (7) is correct for arbitrary dimensions. (Note that e^2 or α carry mass dimensions in arbitrary d.) In the following, we shall see what support we get from the perturbation theory in favour of this assumption.

- Let us first obtain the fermion propagator at order α. To do so, we can set $F = 1$ in equation (6) or (7). A d-dimensional integration can then be carried out and, on setting $d = 4 - 2\epsilon$, we obtain

$$F(p^2) = 1 - \frac{e^2\xi}{(4\pi)^{2-\epsilon}} \frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} (p^2)^{-\epsilon} \tag{8}$$

  which is the *exact* one-loop result in arbitrary dimensions [15]. At the one-loop level, dimensional integration shows that the $\mathcal{A}$-term is identically zero (see the appendix). In fact, interestingly, the $\mathcal{A}$-term vanishes at all orders in the perturbation theory. We again leave the details to the appendix. This means that the $\mathcal{A}$-term in equation (6) can safely be discarded once and for all. This is one of the significant new results of our paper and it is completely independent of any truncations of equation (6).
- Now that we know the one-loop result, we can get the higher powers of $(\alpha\xi)$ simply by successive substitution. Thus putting the above answer back in equation (7) we get the two-loop result. A simple exercise gives

$$\begin{aligned} F(p^2) = 1 &- \frac{e^2\xi}{(4\pi)^{2-\epsilon}} \frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} (p^2)^{-\epsilon} \\ &- \frac{e^4\xi^2}{(4\pi)^{4-2\epsilon}} \frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)\Gamma(2\epsilon)}{\Gamma(2-3\epsilon)} (p^2)^{-2\epsilon} \end{aligned} \tag{9}$$

  which encapsulates the $(\alpha\xi)^2$ term in arbitrary dimensions. A complete calculation of the two-loop fermion propagator can be found in [17]. We can keep repeating the procedure and we shall get exact leading gauge terms at all orders, i.e., terms of the type $\alpha\xi$, $(\alpha\xi)^2$, $(\alpha\xi)^3$, $(\alpha\xi)^4$, and so on. Each time, the integral involved is similar to the one previously evaluated. Therefore, it is rather easy to know the coefficient of the term $(\alpha\xi)^n$ for arbitrarily large values of n.

One should note that although the above procedure accounts for all the terms of the type $(\alpha\xi)^n$ at an arbitrary order of approximation, it does not incorporate non-leading gauge terms, e.g., the ones proportional to α^2, $\alpha^3\xi$, etc. Can a simple extension of equation (7) achieve this? We attempt to answer this question in the following subsection.

2.2. The complete propagator

It was pointed out in [11] that in four dimensions it is rather simple to generalize equation (7) to incorporate the non-leading gauge terms. Therefore, [11] demonstrates that

$$\frac{1}{F(p^2)} = 1 + (4\pi)^2\gamma \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{F(k^2)}{F(p^2)}\frac{k\cdot q}{k^2q^4} \tag{10}$$

where $\gamma = \gamma_0 + \alpha\xi/4\pi$ (γ_0 is gauge independent and, in the perturbation theory, is of $\mathcal{O}(\alpha^2)$), yields the correct gauge covariance of the fermion propagator in four dimensions in perfect agreement with the rules of the LKFT [2, 3].

The question we raised was whether a similar equation also yields the correct covariance properties in arbitrary dimensions. As γ becomes dimensionful in arbitrary d, a simple extension of this equation might read

$$\frac{1}{F(p^2)} = 1 + (4\pi)^2 \int \frac{\mathrm{d}^dk}{(2\pi)^d}\gamma_d(k^2, p^2, q^2)\frac{F(k^2)}{F(p^2)}\frac{k\cdot q}{k^2q^4} \tag{11}$$

where the function $\gamma_d(k^2, p^2, q^2)$ necessarily depends upon d. Therefore, we can write it as follows:

$$\gamma_d(k^2, p^2, q^2) = \frac{\alpha\xi}{4\pi} + \lambda_d(k^2, p^2, q^2). \tag{12}$$

(The functions $\gamma_d(k^2, p^2, q^2)$ and $\lambda_d(k^2, p^2, q^2)$ acquire mass dimensions $m^{(4-d)}$ like α.) In four dimensions, $\gamma_d(k^2, p^2, q^2)$ is nothing but the anomalous dimension and according to [10] $\gamma_4 = \gamma = a\xi - \frac{3}{2}a^2 + \frac{3}{2}a^3 + \mathcal{O}(a^4)$. Therefore, we can identify

$$\lambda_4(k^2, p^2, q^2) = \lambda_4 = -\tfrac{3}{2}a^2 + \tfrac{3}{2}a^3 + \mathcal{O}(a^4). \tag{13}$$

In the spirit of four dimensions, we seek a $\lambda_d(k^2, p^2, q^2)$ that is independent of the covariant gauge parameter in arbitrary dimensions d. Thus we have squeezed all the information contained in the full fermion–boson vertex (projected onto the fermion propagator) into $\lambda_d(k^2, p^2, q^2)$. Whereas knowing the full vertex at every order of α and thus calculating the fermion propagator is a formidable task, we look for some simple parametric form of the function $\lambda_d(k^2, p^2, q^2)$ which would be sufficient to capture the correct gauge covariance of the propagator at all orders in arbitrary dimensions. As an example, we put to test the following form:

$$\lambda_d(k^2, p^2, q^2) = \alpha^2\left[\frac{\lambda_1^d}{(k^2)^{2-d/2}} + \frac{\lambda_1'^d}{(p^2)^{2-d/2}}\right] + \alpha^3\left[\frac{\lambda_2^d}{(k^2)^{2(2-d/2)}} + \frac{\lambda_2'^d}{(p^2)^{2(2-d/2)}}\right] + \mathcal{O}(\alpha^4) \tag{14}$$

where we expect the constants λ_i and λ_i' all to be independent of the gauge parameter ξ. (Note that we have avoided including q^2 dependence in (29). Our guiding principle was to choose a form as simple as possible without spoiling the objective of the exercise. We could readily have introduced mixed terms of the type $1/(k^2p^2)^{2-d/2}$, $\Delta^{-2+d/2}$ or even terms incorporating q^2 to the appropriate power. It is perfectly conceivable that they can arise in more complicated topological contributions to the vertex function but we have strong hints that they are absent in the asymptotic regime $k^2 \gg p^2$ which we shall be considering shortly.) Equations (11), (12) and (14) form the main equations of this section. We point out that for $d = 4$, the momentum dependence of the gamma function disappears as expected and we can identify, λ_i^4 and $\lambda_i'^4$ by making a direct comparison with equation (13).

Before we try to see if equations (11), (12) and (14) give the propagator equation with correct properties in arbitrary dimensions, let us clarify its status. This equation does not determine the values of λ_i and λ_i' and such like. However, given such constants, one can determine the propagator in an arbitrary covariant gauge. This is similar to what the LKFT specify: given the fermion propagator in one particular gauge, they determine what the propagator will be in an arbitrary covariant gauge. However, the LKFT are written in the coordinate space and the Fourier transforms involved can only be performed easily in certain very limited number of special cases. On the other hand, solving equation (11) is much simpler, as we have demonstrated early on.

We do not attempt here to prove the validity of equations (11), (12) and (14) in all dimensions, but we expect that some simple parametric form of $\lambda_d(k^2, p^2, q^2)$, like ours, is able to give the correct gauge dependence of the full fermion propagator. For now we only verify that this form does indeed yield the correct gauge dependence of the fermion propagator for $d = 3$ at least up to three loops in perturbation theory. A straightforward calculation reveals that if we select

$$\lambda_1^3 = \frac{3}{16}\left(\frac{7}{3} - \frac{\pi^2}{4}\right) \qquad \lambda_1'^3 = 0$$

equation (11) gives the following result until $\mathcal{O}(\alpha^3)$:

$$F(p^2) = 1 - \frac{\pi}{4}\frac{(\alpha\xi)}{p} + \frac{1}{4}\frac{(\alpha\xi)^2}{p^2} - \frac{3}{4}\left(\frac{7}{3} - \frac{\pi^2}{4}\right)\frac{\alpha^2}{p^2} - \pi^2\lambda_2'^3\frac{\alpha^3}{p^3}. \tag{15}$$

Comparing it with the findings in [18] which gives the massless fermion propagator to two loops in quenched QED3, we confirm our truncation to this order. Moreover, [19] tells us that $\mathcal{O}(\alpha^3)$ term must be independent of the gauge parameter. This is again in perfect agreement with our assumption that all the λ_i and λ_i' (in this case $\lambda_2'^3$) must be independent of the gauge parameter.

This example provides a verification that our proposal for the λ_d function works for QED_3 up to three loops in addition to working for QED_4 at all orders. The λ_i^3 and $\lambda_i'^3$ are gauge-independent quantities as we had claimed. Moreover, we do not get any terms proportional to $(\alpha\xi)^3$, $\alpha^3\xi^2$ and $\alpha^3\xi$, as already expected from the LKFT [19]. It would be worth checking if this simple form of the function $\lambda_d(k^2, p^2, q^2)$ could yield the correct form of the fermion propagator in three dimensions at higher orders and in other dimensions but we do not address this issue in this paper. In the following section, we discuss the constraints imposed by our proposal and the perturbation theory for the fermion propagator on the 3-point fermion–boson vertex.

3. Asymptotic constraints

3.1. Constraints from the SDE on $S(p)$

After demonstrating with examples that equations (11), (12) and (14) may account for the fermion propagator, we should separate the rest of the terms in equation (6), which start off at order α^2. These ought then to conspire against each other to give zero in the asymptotic regime at the very least, namely

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{F(k^2)}{k^2q^2}\left\{\mathcal{B}(k^2,p^2)\frac{1}{q^2}[-2\Delta^2(k^2+p^2)] + \frac{\lambda_d(k^2,p^2,q^2)}{\alpha F(p^2)}\frac{p^2}{q^2}(k^2-k\cdot p)\right.$$
$$+\tau_2(k^2,p^2,q^2)[-\Delta^2(k^2+p^2)] + \tau_3(k^2,p^2,q^2)[-2\Delta^2+(1-d)q^2k\cdot p]$$
$$\left.+\tau_6(k^2,p^2,q^2)[(1-d)(k^2-p^2)k\cdot p] + \tau_8(k^2,p^2,q^2)[(d-2)\Delta^2]\right\} = 0. \tag{16}$$

This equation naturally places a constraint on the choice of the transverse part of the fermion–boson vertex. Any *ansatz* for the transverse vertex must ensure that this condition is satisfied for $k^2 \gg p^2$ anyhow.

In principle, knowledge of the transverse vertex at the lowest order in arbitrary dimensions and gauge [15], allows us to verify this condition at that level of approximation. But the expressions for the τ_i are complicated and lengthy, rendering the exercise highly non-trivial. However, we have managed to check that at least for $d = 3$, this condition is indeed satisfied. In a previous work, it was checked also for $d = 4$ [11].

3.2. Perturbation theory constraints

We know that the perturbation theory is the only calculational scheme which incorporates the key features of a gauge field theory such as gauge identities at every order of approximation naturally. Therefore, we stand a better chance in retaining these essential properties if every nonperturbative *ansatz* of the 3-point vertex in truncating the tower of Schwinger–Dyson equations makes sure that it reduces to its perturbative Feynman expansion when the coupling involved is weak. Whereas for arbitrary momenta of the fermions and photon, this expansion can be very complicated, blurring our view of its possible nonperturbative forms, certain asymptotic regimes of momentum can turn out to be a better guide in this respect. In general, the τ_i are explicit functions of the angle between k and p. Only in certain asymptotic limits does this dependence disappear. One such limit is when momentum k^2 in one of the fermion legs is much greater than the one in the other leg, namely p^2. A recent calculation of Davydychev *et al* of the τ_i at the one-loop level in arbitrary dimensions enables us to take this limit. That is what we do next. One such integral that is somewhat more difficult to evaluate in this limit is [20]

$$J(4-2\epsilon) = \int \mathrm{d}^d\omega\,\frac{1}{\omega^2(k-\omega)^2(p-\omega)^2} = -\frac{2\pi^{2-\epsilon}\mathrm{i}^{1+2\epsilon}}{(-k^2p^2q^2)^\epsilon}\frac{\Gamma^2(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}$$
$$\times\left[\frac{(k^2p^2)^\epsilon}{(k^2+p^2-q^2)}\,{}_2F_1\left(1,\frac{1}{2};\frac{3}{2}-\epsilon;\frac{4\Delta^2}{(k^2+p^2-q^2)^2}\right)\right.$$
$$+\frac{(k^2q^2)^\epsilon}{(k^2+q^2-p^2)}\,{}_2F_1\left(1,\frac{1}{2};\frac{3}{2}-\epsilon;\frac{4\Delta^2}{(k^2+q^2-p^2)^2}\right)$$
$$+\frac{(q^2p^2)^\epsilon}{(q^2+p^2-k^2)}\,{}_2F_1\left(1,\frac{1}{2};\frac{3}{2}-\epsilon;\frac{4\Delta^2}{(q^2+p^2-k^2)^2}\right)$$
$$\left.-\frac{\pi\Gamma(2-2\epsilon)}{\Gamma^2(1-\epsilon)}[-4\Delta^2]^{-\frac{1}{2}+\epsilon}\theta_{123}\right] \tag{17}$$

where

$$\theta_{123} = \theta(k^2 + p^2 - q^2)\theta(k^2 + q^2 - p^2)\theta(q^2 + p^2 - k^2).$$

In the Euclidean limit $k^2 \gg p^2$, the leading term would be independent of the angle between k and p. Therefore, the most convenient way to proceed is to choose a particular angle for which it is easier to take the limit. In order to confirm our results, we calculate J for two angles between k and p, namely 0 and $\pi/2$, and expectedly arrive at the same result:

$$J(4-2\epsilon) = -\mathrm{i}\pi^{2-\epsilon}\frac{\Gamma^2(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)k^2}((p^2)^{-\epsilon} - (k^2)^{-\epsilon}). \tag{18}$$

In three and four dimensions, this expression reduces to the already known results [11, 18]. Further, it can now be related to the 2-point integral [20]:

$$K(k^2; 4-2\epsilon) = \int \mathrm{d}^d\omega \frac{1}{\omega^2(k-\omega)^2} = \mathrm{i}\pi^{2-\epsilon}\frac{\Gamma^2(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}(k^2)^{-\epsilon}. \tag{19}$$

Therefore, the 3-point integral can be written as the difference of two 2-point integrals in the limit when $k^2 \gg p^2$:

$$J(4-2\epsilon) = \frac{1}{k^2}[K(k^2; 4-2\epsilon) - K(p^2; 4-2\epsilon)]. \tag{20}$$

It is interesting to note that the lightlike limit of the vertex function exhibits precisely the same form, namely $[K(k^2) - K(p^2)]/(k^2 - p^2)$, as has been pointed out in [21, 22].

We may now anticipate that in this limit, the 3-point vertex will bear a simple relationship with the 2-point propagator. However, for the actual evaluation of the transverse vertex or the τ_i in this limit, we have to evaluate J to the next order of approximation. On doing so, we obtain the following expression for the τ_i:

$$\tau_2(k^2, p^2) = \frac{e^2}{(4\pi)^{2-\epsilon}}\frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}\left[\frac{-1+(2-\epsilon)\xi}{3-2\epsilon}\right]\frac{1}{k^4}((p^2)^{-\epsilon} - (k^2)^{-\epsilon})$$

$$\tau_3(k^2, p^2) = \frac{e^2}{2(4\pi)^{2-\epsilon}}\frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}\left[\frac{2(1-\epsilon)-(2-\epsilon)\xi}{3-2\epsilon}\right]\frac{1}{k^2}((p^2)^{-\epsilon} - (k^2)^{-\epsilon})$$

$$\tau_6(k^2, p^2) = \frac{e^2}{2(4\pi)^{2-\epsilon}}\frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}\left[\frac{2(1-\epsilon)-(1-\epsilon)\xi}{3-2\epsilon}\right]\frac{1}{k^2}((p^2)^{-\epsilon} - (k^2)^{-\epsilon})$$

$$\tau_8(k^2, p^2) = \frac{e^2}{(4\pi)^{2-\epsilon}}\frac{\Gamma^2(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}[1+\epsilon\xi]\frac{1}{k^2}((p^2)^{-\epsilon} - (k^2)^{-\epsilon}).$$

Any *ansatz* for the transverse vertex must reproduce these results in the weak coupling regime and in the asymptotic limit of momenta considered. As a check, these expressions nicely reproduce the results quoted in [11, 18] for the special cases of $d = 4$ and 3 respectively. The τ_i can now be written in terms of the fermion propagator function F, equation (8), just like the coefficients of the longitudinal vertex. In the asymptotic limit mentioned above, a straightforward exercise yields

$$\tau_2(k^2, p^2) = \frac{1-(2-\epsilon)\xi}{3-2\epsilon}\frac{1}{k^4}\left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right] \tag{21}$$

$$\tau_3(k^2, p^2) = -\frac{2(1-\epsilon)-(2-\epsilon)\xi}{2(3-2\epsilon)}\frac{1}{k^2}\left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right] \tag{22}$$

$$\tau_6(k^2, p^2) = -\frac{(1-\epsilon)(2-\xi)}{2(3-2\epsilon)}\frac{1}{k^2}\left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right] \tag{23}$$

$$\tau_8(k^2, p^2) = -\frac{1}{1-\epsilon}\frac{1}{k^2}\left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right]. \tag{24}$$

Thus the τ_i come out to be related to the fermion propagator in a strikingly similar manner as the coefficients of the longitudinal vertex are related to it. It will be intriguing if these relations remain intact at higher orders in the perturbation theory. Taking into account the tensor structures T_i^μ, one can readily see that $\tau_2(k^2, p^2)$ and $\tau_8(k^2, p^2)$ provide *subleading* contributions to the transverse vertex. Therefore, the complete transverse vertex *in this limit* can be written as

$$\Gamma_T^\mu(k, p) = \frac{e^2\xi}{2(4\pi)^{2-\epsilon}}\frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}\frac{1}{k^2}((p^2)^{-\epsilon} - (k^2)^{-\epsilon})[-k^2\gamma^\mu + k^\mu \not{k}] \tag{25}$$

which is a result of a magic cancellation of some d-dependent terms in the expressions for τ_3 and τ_6. One can readily verify that in particular cases of four and three dimensions respectively, this expression reproduces well-known results [11, 18]. In terms of the fermion propagator, this expression can be written as

$$\Gamma_T^\mu(k, p) = \frac{\xi}{2k^2}\left[\frac{1}{F(k^2)} - \frac{1}{F(p^2)}\right][k^2\gamma^\mu - k^\mu \not{k}]. \tag{26}$$

This is very much like the longitudinal vertex written in terms of the fermion propagator, guided by the WGTI. It will be interesting to see if this simple relation survives the complications of higher orders of the perturbation theory. Knowledge of the transverse vertex for an arbitrary regime of fermion momenta in its nonperturbative form is a highly nontrivial task. So far, only for three dimensions has such a vertex been proposed [14]. However, we believe that the perturbation theory at higher orders along with gauge identities should supply better insight in arbitrary dimensions.

4. Conclusions

The nonperturbative fermion propagator and the fermion–boson vertex are in principle determined from the corresponding Schwinger–Dyson equations, although in practice, solving these equations exactly is a prohibitively difficult task. However, covariance gauge identities impose strict constraints on how these Green functions are related to one another and how they evolve under a variation of gauge. The perturbation theory too is an important guide where the systematic scheme of approximation ensures that these identities are satisfied identically at every order of α. Based upon these ingredients, we have analysed the fermion propagator and the fermion–boson vertex in a nonperturbative fashion in arbitrary dimensions. We have proposed a simple equation for the fermion propagator written in the momentum space which seems a likely candidate to capture the correct gauge covariance of this Green function, at asymptotic values in any event. As is written in the momentum space, it is much easier to extract information compared to the coordinate space LKFT. Guided by this equation and the perturbation theory, we have arrived at (asymptotic) constraints on the fermion–boson vertex. We hope that higher order perturbative calculations and the LKFT for the vertex itself could help us further pin down the possible nonperturbative forms of these Green functions in arbitrary dimensions. All this is for the future.

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Appendix A

At the one-loop level, the $\mathcal{A}$-term in equation (6) corresponds to setting $F(k^2) = 1$ and evaluating the integral:

$$I \equiv \int \frac{\mathrm{d}^d k}{k^2 q^4} [(1-d)q^2 k \cdot p - 2\Delta^2]. \tag{A.1}$$

Re-writing $k \cdot p = (k^2 + p^2 - q^2)/2$ and $\Delta^2 = (q^4 + k^4 + p^4 - 2k^2 p^2 - 2q^2(k^2 + p^2))/4$, and using the fact that the d-dimensional integration without the presence of external momenta is zero, we simply arrive at the following expression:

$$I = \frac{p^2}{2} \left[(3-d) \int \frac{\mathrm{d}^d k}{k^2 q^2} - p^2 \int \frac{\mathrm{d}^d k}{k^2 q^4} \right]. \tag{A.2}$$

Adopting the notation

$$J(\nu_1, \nu_2) = \int \frac{\mathrm{d}^d k}{(k^2)^{\nu_1} (q^2)^{\nu_2}} \tag{A.3}$$

we can now make use of the recursion relation [23],

$$J(\nu_1, \nu_2 + 1) = \frac{-1}{\nu_2 p^2} [(d - 2\nu_1 - \nu_2) J(\nu_1, \nu_2) - J(\nu_1 - 1, \nu_2 + 1)]$$

to deduce that I is identically zero in arbitrary dimensions. What happens to all orders? We now have to evaluate the integral $\mathcal{I}$:

$$\mathcal{I} \equiv \frac{1}{2} \int \frac{\mathrm{d}^d k}{k^2 q^4} \left[1 + \frac{F(k^2)}{F(p^2)} \right] [(1-d)q^2 k \cdot p - 2\Delta^2] \tag{A.4}$$

which reduces to I at the one-loop level. We proceed by dividing the d-dimensional integral into radial and angular integrals so that $\mathrm{d}^d k = \mathrm{d}k\, k^{d-1} \sin^{d-2}\theta\, \mathrm{d}\theta\, \Omega_{d-2}$, where $\Omega_{d-2} = 2\pi^{(d-1)/2} / \Gamma((d-1)/2)$. The angular integral to be evaluated is

$$\hat{I} = \int_0^\pi \mathrm{d}\theta \frac{\sin^{d-2}\theta}{q^4} [2(d-2)k^2 p^2 \cos^2\theta + (1-d)kp(k^2 + p^2)\cos\theta + 2k^2 p^2]. \tag{A.5}$$

Note that the integrand is proportional to the full derivative $(\mathrm{d}/\mathrm{d}\theta)[\sin^{(d-1)}\theta/q^2]$. Thus the result is zero because $\sin 0 = \sin\pi = 0$ at the integration limits.

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